On the Steklov-Lyapunov case of the rigid body motion.

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Abstract

We construct a Poisson map between manifolds with linear Poisson brackets corresponding to the two samples of Lie algebra $e(3)$. Using this map we establish equivalence of the Steklov-Lyapunov system and the motion of a particle on the surface of the sphere under the influence of the fourth order potential. To study separation of variables for the Steklov case on the Lie algebra $so(4)$ we use the twisted Poisson map between the bi-Hamiltonian manifolds $e(3)$ and $so(4)$.

PACS numbers: 02.30.Ik, 02.30.Uu, 02.30.Zz, 02.40.Yy, 45.30.+s

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1 Introduction

A standard form of the Kirchhoff equations is the following:

$$\dot{M} = M \times \Omega + p \times U, \qquad \dot{p} = p \times \Omega, \tag{1.1}$$

where $X \times Y$ stands for the vector product of three-dimensional vectors [6].

These equations describe the motion of a rigid body in the ideal incompressible fluid if two vectors M and p are the impulsive momentum and the impulsive force while the vectors Ω and U are the angular and linear velocities of the body. All these vectors in $\mathbb{R}^3$ are expressed in the body frame attached to the body [6].

It is known [11] that the system (1.1) is Hamiltonian with respect to the Lie–Poisson bracket of the Lie algebra e(3) of the Lie group $E(3)$ of Euclidean motions of $\mathbb{R}^3$, i.e. with respect to the Poisson bracket

$$\{M_i, M_j\}_1 = \varepsilon_{ijk} M_k, \quad \{M_i, p_j\}_1 = \varepsilon_{ijk} p_k, \quad \{p_i, p_j\}_1 = 0\,, \tag{1.2}$$

where ε_{ijk} is the totally skew-symmetric tensor. Here and below we identify dual space $e^*(3)$ with $e(3)$ using the standard inner product [11] .

The Hamiltonian equations of motion for an arbitrary Hamilton function $H = H(p, M)$ in the bracket (1.2) read

$$\dot{M} = M \times \nabla_M H + p \times \nabla_p H, \qquad \dot{p} = p \times \nabla_M H. \tag{1.3}$$

This Euler's equations on $e^*(3)$ coincides with (1.1) if

$$H(p, M) = \langle M, \mathbf{A}_1 M\rangle + \langle M, \mathbf{A}_2 p\rangle + \langle p, \mathbf{A}_3 p\rangle.$$

Here $\mathbf{A}_k$ are special numerical matrices and $\langle ., .\rangle$ stands for the standard scalar product in $\mathbb{R}^3$ [11].

The Poisson bracket (1.2) has two Casimir function

$$\mathcal{A} = |p|^2, \qquad \mathcal{B} = \langle p, M\rangle\,. \tag{1.4}$$

which are therefore integrals of motion for (1.1) in involution with Hamiltonian $H(p, M)$ and with any other function on the phase space.

The Steklov-Lyapunov case of the rigid body motion (the Steklov-Lyapunov system, for brevity), is characterized by the following diagonal matrices

$$\mathbf{A} = \mathrm{diag}(a_1, a_2, a_3), \qquad \mathbf{C} = \mathrm{diag}(a_2 - a_3, a_3 - a_1, a_1 - a_2)\,. \tag{1.5}$$

In [14] Steklov found a Hamilton function

$$4H_S(p, M) = \langle M, \mathbf{A}M\rangle + 2\langle M, \mathbf{A}^{\vee} p\rangle + \langle \mathbf{A}p, \mathbf{C}^2 p\rangle \tag{1.6}$$

for which equations (1.3) possess a fourth additional integral. Here wedge denotes adjoint matrix, i.e. cofactor matrix. In our case it reads $\mathbf{A}^{\vee} = (\det \mathbf{A})\, \mathbf{A}^{-1}$.

Later Lyapunov [8] independently discovered an integrable case of the Kirchhoff equations whose Hamiltonian

$$4H_L(p, M) = \langle M, M\rangle - 2\langle M, \mathbf{A}p\rangle + \langle p, \mathbf{C}^2 p\rangle \tag{1.7}$$

is a linear combination of the Steklov integrals [14] and the Casimir functions (1.4).

The Lax matrices for the Steklov-Lyapunov system may be extracted from the Kötter work [1, 2]. Namely, in [7] Kötter introduce two vectors ℓ and m

$$\ell(\lambda) = \mathbf{W}\left(\frac{1}{2}(M - \mathbf{B}p) + \lambda\, p\right), \qquad m(\lambda) = \mathbf{W}^{\vee} p\,, \tag{1.8}$$

where $\mathbf{W}$ and $\mathbf{B}$ are diagonal matrices with the following entries

$$\mathbf{W}_{ii} = \sqrt{\lambda - a_i}, \qquad \mathbf{B}_{ii} = \sum_{j,k=1}^{n=3} |\varepsilon_{ijk}|\, a_k\,. \tag{1.9}$$

Here λ is auxiliary variable (spectral parameter) and functions $\mathbf{W}_{ii} = \sqrt{\lambda - a_i}$ can be considered as basic elliptic functions (see [1, 2]). The equations of motion

$$\frac{d}{dt}\ell(\lambda) \equiv \{H_L, \ell\}_1 = m(\lambda) \times \ell(\lambda)\,, \tag{1.10}$$

may be rewritten in the Lax form

$$\frac{d}{dt}\mathscr{L}_e(\lambda) = [\mathscr{M}_e(\lambda), \mathscr{L}_e(\lambda)] \tag{1.11}$$

using two Lax matrices

$$\mathscr{L}_e(\lambda) = \sum_{i=1}^{3} \ell_i(\lambda)\sigma_i\,, \qquad \mathscr{M}_e(\lambda) = \sum_{i=1}^{3} m_i(\lambda)\sigma_i\,, \tag{1.12}$$

where σ_i are the Pauli matrices.

For an actual integration of the corresponding Hamiltonian flow in terms of elliptic functions see [7], and for a more modern account [1, 2]. In fact in the known integration procedure we have to use the Lax matrices (1.12) with the spectral parameter λ varying on an elliptic curve.

However, it is known that the Steklov-Lyapunov flow is linearizable on the Jacobian of hyperelliptic curve instead of elliptic curve, similar to the Neumann system [2]. Therefore, we can suppose that the Steklov-Lyapunov system belong to the family of the Stäckel systems [16] and there are the Lax matrices with rational dependence on the spectral parameter.

The desired rational Lax matrices were constructed by Bolsinov and Fedorov by using a special triplets of vectors, which are coordinates on some complicated phase space (see [3] and references within).

The aim of this note is to identify the Steklov-Lyapunov system with an integrable motion of a particle on the surface of the sphere, which is the Stäckel system. This result is the direct sequence of the Kötter separation of variables [7].

2 Separation of variables

If σ_k are 3×3 Pauli matrices, then the Lax matrix $\mathscr{L}_e(\lambda)$ (1.12) looks like

$$\mathscr{L}_e(\lambda) = \begin{pmatrix} 0 & \ell_3 & -\ell_2 \\ -\ell_3 & 0 & \ell_1 \\ \ell_2 & -\ell_1 & 0 \end{pmatrix} \in so(3)\,.$$

The corresponding spectral curve is defined by equation $\det(\mu-\mathscr{L}_e(\lambda))=0$, which is reduced to the following equation

$$\mathcal{C}:\qquad \mu^2+P_3(\lambda)=0\,, \tag{2.1}$$

where

$$P_3(\lambda)=|\ell|^2=\alpha^2\lambda^3+(\beta-\alpha^2\mathrm{tr}\,\mathbf{A})\lambda^2+H_1\,\lambda+H_2\,. \tag{2.2}$$

Coefficients $H_{1,2}$ are linear combination of the Steklov- Lyapunov integrals (1.6-1.7) and the Casimir functions (1.4)

$$H_1=H_L-\frac{\mathcal{B}}{2}\,\mathrm{tr}\,\mathbf{A}+\mathcal{A}\,\mathrm{tr}\,\mathbf{A}^\vee\,,\qquad H_2=-H_S+\frac{\mathcal{B}}{2}\,\mathrm{tr}\,\mathbf{A}^\vee-\mathcal{A}\,\det\,\mathbf{A}\,. \tag{2.3}$$

The separation of variables associated with the curve (2.1) was constructed by Kötter [7] and may be recovered in framework of the modern Sklyanin method [13].

Proposition 1 *Separated variables $u_{1,2}$ associated with the Lax matrix $\mathscr{L}_e(\lambda)$ are poles of the corresponding Baker-Akhiezer function $\mathbf{\Psi}(\lambda)$ with the following dynamical normalization*

$$\boldsymbol{\alpha}=\frac{a}{|(M-\mathbf{B}p)\times p\,|}\,\mathbf{W}\,p,\qquad a\in\mathbf{R}\,. \tag{2.4}$$

The proof consists of direct comparison of the known separated variables [7] with poles of the Baker-Akhiezer function $\mathbf{\Psi}(\lambda)$, which is an eigenvector of the Lax matrix

$$\mathscr{L}_e(\lambda)\mathbf{\Psi}(\lambda)=\mu\mathbf{\Psi}(\lambda)\,. \tag{2.5}$$

Since an eigenvector is defined up to a scalar factor one has to fix a normalization of $\mathbf{\Psi}(\lambda)$ imposing a linear constraint

$$\langle\boldsymbol{\alpha},\mathbf{\Psi}(\lambda)\rangle=1. \tag{2.6}$$

In our case $\mathscr{L}_e(\lambda)\in so(3)$ and excluding μ from (2.5-2.6) we derive that poles u_k of $\mathbf{\Psi}(\lambda)$ are roots of the following equation

$$\big\langle\boldsymbol{\alpha}\times\ell,\boldsymbol{\alpha}\times\ell\big\rangle=0\,,$$

where ℓ is the Kötter vector (1.8).

Inserting $\boldsymbol{\alpha}$ (2.4) in this equations and dividing it on polynomial $\det(\lambda-\mathbf{A})$ one can define the separated variables $u_{1,2}$ as roots of the following function

$$e(\lambda)=\frac{(\lambda-u_1)(\lambda-u_2)}{\det(\lambda-\mathbf{A})}=\frac{\langle\boldsymbol{\alpha}\times\ell,\boldsymbol{\alpha}\times\ell\rangle}{\det(\lambda-\mathbf{A})}\equiv\sum_{i=1}^{n=3}\frac{x_i^2}{\lambda-a_i}=0\,, \tag{2.7}$$

where vector x is given by

$$x=a\frac{(M-\mathbf{B}p)\times p}{|(M-\mathbf{B}p)\times p\,|}\,,\qquad a\in\mathbb{R}\,.$$

The equation (2.7) coincides with definition of the separated variables from [7], where Kötter also proved that initial equations of motion (1.1) can be written in the form

$$\dot u_1=\frac{2\sqrt{P_3(u_1)\det(\mathbf{A}-u_1)}}{u_1-u_2},\qquad \dot u_2=-\frac{2\sqrt{P_3(u_2)\det(\mathbf{A}-u_2)}}{u_1-u_2} \tag{2.8}$$

and then he integrated these equations using Abel-Jacobi inversion theorem.

Proposition 2 *The Kötter variables $u_{1,2}$ (2.7) and momenta $v_{1,2}$ defined by*

$$v_j = -\frac{1}{2a^2}\,\{H_1, e(\lambda)\}\Big|_{\lambda=u_j} \qquad j=1,2, \tag{2.9}$$

are canonical variables

$$\{u_1,u_2\}_1 = \{v_1,v_2\}_1 = 0, \qquad \{v_j,u_k\}_1 = \delta_{jk}\,,$$

which satisfy to the following separated equations

$$v_j^2 + \frac{P_3(u_j)}{\det(u_j-\mathbf{A})} = 0\,, \qquad j=1,2. \tag{2.10}$$

The proof is straightforward.

As sequence of (2.10), in canonical variables $u_{1,2}$ and $v_{1,2}$ integrals of motion $H_{1,2}$ are the Stäckel integrals

$$H_j = \sum_{k=1}^{2} \mathcal{S}^{-1}_{jk}\Big(\varphi(u_k)\,v_k^2 - U(u_k)\Big)\,, \qquad j=1,2, \tag{2.11}$$

where

$$\varphi(\lambda) = \det(\mathbf{A}-\lambda)\,, \qquad U(\lambda) = \alpha^2\lambda^3 + (\beta-\alpha^2\mathrm{tr}\,\mathbf{A})\lambda^2\,,$$

and $\mathcal{S}$ is the Stäckel matrix

$$\mathcal{S} = \begin{pmatrix} u_1 & u_2 \\ 1 & 1 \end{pmatrix}. \tag{2.12}$$

This matrix is 2×2 block of the transpose Brill-Noether matrix $\mathcal{U}_{\mathcal{C}}$

$$\mathcal{U}_{\mathcal{C}} = \begin{pmatrix} u_1^3 & u_1^2 & u_1 & 1 \\ u_2^3 & u_2^2 & u_2 & 1 \end{pmatrix}, \tag{2.13}$$

which determines the Abel-Jacobi map on Jacobian of hyperelliptic curve $\mathcal{C}$ (2.1) completely.

In our case the Stäckel matrix $\mathcal{S}$ (2.12) is the lowest block of the transpose Brill-Noether matrix $\mathcal{U}_{\mathcal{C}}$ (2.13)) and, therefore, there are canonical coordinates in which equations of motion (2.8) are the Newton equations [16, 17].

In the next section we discuss this canonical change of variables $(p,M)\to(x,J)$ which transforms the Kirchhoff equations (1.1) to the Newton equations in detail.

3 The Poisson map

The Poisson manifold is a smooth manifold $\mathcal{M}$ endowed with the Poisson brackets $\{.\,,.\}_{\mathcal{M}}$. If $\mathcal{M}$ and $\mathcal{N}$ are Poisson manifolds, a smooth map $f:\mathcal{M}\to\mathcal{N}$ is called a Poisson map provided that it preserves Poisson brackets, i.e.

$$f^*\Big\{\varphi,\psi\Big\}_{\mathcal{N}} = \Big\{f^*\varphi, f^*\psi\Big\}_{\mathcal{M}}$$

for all $\varphi,\psi\in C^\infty(\mathcal{N})$. Here $f^*\varphi = \varphi\circ f$ is a lifting of the function $\varphi\in C^\infty(\mathcal{N})$ on $\mathcal{M}$.

Below we deal with linear Poisson brackets corresponding to the two samples of Lie algebra $e(3)$ and to one sample of $so(4)$ algebra, i.e. with homomorphisms of these Lie algebras. For brevity we will use the same notations both for the Poisson manifolds and the Lie algebras.

If $\mathcal{M}$ coincides with $\mathcal{N}$, the Poisson maps are called *canonical transformations*. The problem of complete efficient description of all nonlinear canonical transformations is unsolvable. The reason is that for any function $f(\mathbf{x}_1,\ldots,x_n)$ the flow defined by ODEs $\dot{x}_i = \{f\,,x_1,\ldots,x_n\}$ yields a one-parameter group of canonical transformations. However one can investigate some interesting subgroups of nonlinear canonical transformations.

Let x and J are coordinates on the Lie algebra $e(3)$ with the standard Lie-Poisson brackets

$$\bigl\{J_i\,,J_j\bigr\}_1' = \varepsilon_{ijk}J_k\,,\qquad \bigl\{J_i\,,x_j\bigr\}_1' = \varepsilon_{ijk}x_k\,,\qquad \bigl\{x_i\,,x_j\bigr\}_1' = 0\,. \tag{3.1}$$

The brackets (3.1) respect two Casimir elements

$$A = |x|^2 \equiv \sum_{k=1}^{n=3} x_k^2,\qquad B = \langle x, J\rangle \equiv \sum_{k=1}^{n=3} x_k J_k\,. \tag{3.2}$$

Fixing values of the Casimir elements one gets a generic symplectic leaf of $e(3)$

$$\mathcal{O}_{ab} = \{x, J\ :\ A = a^2,\quad B = b\}\,, \tag{3.3}$$

which is topologically equivalent to cotangent bundle T^*S^2 of the sphere

$$S^2 = \{x \in \mathbb{R}^3, |x| = a\}.$$

Symplectic structure of $\mathcal{O}_{ab}$ is different from the standard symplectic structure on T^*S^2 by the magnetic term proportional to b [11].

If $b = 0$ there is standard Poisson map $T^*S^2 \to e(3)$

$$\rho:\quad (\pi, x) \to J = \pi \times x, \tag{3.4}$$

where $\pi \in \mathbb{R}^3$ is conjugated to x momenta

$$\{\pi_i, x_j\} = \delta_{ij},\qquad \text{and}\qquad \langle x, \pi\rangle = 0. \tag{3.5}$$

Let us consider classical counterpart of the Fourier transformation $f : T^*S^2 \to T^*S^2$ defined by

$$f:\quad (x, \pi) \to (-\pi, x)\,. \tag{3.6}$$

This symplectic mapping may be lifted to the Poisson mapping $\widetilde{f} : e(3) \to e(3)$

$$\widetilde{f}:\quad (p, M) \to \left(x = \sqrt{a^2 - \frac{b^2}{|M|^2}}\,\frac{M\times p}{|\,M\times p\,|} + b\frac{M}{|M|^2}\,,\qquad J = M \right). \tag{3.7}$$

Its inverse mapping looks like

$$\widetilde{f}^{-1}:\quad (x, J) \to \left(p = -\sqrt{\alpha^2 - \frac{\beta^2}{|J|^2}}\,\frac{J\times x}{|\,J\times x\,|} + \beta\frac{J}{|J|^2}\,,\qquad M = J \right). \tag{3.8}$$

The maps $\widetilde{f}$ and $\widetilde{f}^{-1}$ couple coordinates (x,J) on algebra $e(3)$ (3.1) with the following values of the Casimir functions (3.2)

$$A = a^2, \qquad B = b$$

and coordinates (p,M) on another sample of $e(3)$ (1.2) with the following values of the Casimir elements (1.4)

$$\mathcal{A} = \alpha^2\,, \qquad \mathcal{B} = \beta.$$

The symplectic map f (3.6) on T^*S^2 is easy generalized:

$$f_g: \qquad (x,\pi) \to (-\pi + g(x), x)$$

if $g(x)$ is a function on x such that

$$\langle x, g(x)\rangle = 0.$$

For instance we can put $g(x) = x \times \mathbf{B}x$, where $\mathbf{B}$ is an arbitrary numerical matrix.

For brevity the lifting of this symplectic map f_g to the Poisson map we present at the special case $\mathbf{B} = \mathrm{diag}(b_1, b_2, b_3)$ and $b = 0$ only. It will be enough in order to identify the Steklov-Lyapunov system with another integrable systems on $e(3)$ [19].

Proposition 3 *Let* $\mathbf{B}$ *and* $\mathbf{C}$ *are numerical diagonal matrices with the following entries*

$$\mathbf{B}_{ii} = b_i\,, \quad \mathbf{C}_{ii} = \sum_{j,k=1}^{n=3} \varepsilon_{ijk}\, b_j\,, \qquad b_i \in \mathbb{R}. \tag{3.9}$$

The mapping $\widetilde{f}_g : (p,M) \to (x,J)$ *defined by*

$$x = a\frac{(M - \mathbf{B}p)\times p}{|(M - \mathbf{B}p)\times p\,|}, \qquad J = M + a^{-2}\mathbf{C}\Big[\,x, x\times p\,\Big]_+\,, \tag{3.10}$$

is a Poisson map $\widetilde{f}_g : e(3) \to e(3)$ *such that*

$$A = |x|^2 = a^2, \qquad B = \langle x, J\rangle = 0.$$

Here $[y,z]_+$ *is an "anticommutator" of two vectors* y *and* z *defined by*

$$[y,z]_{+i} = \sum_{j,k=1}^{n=3} |\varepsilon_{ijk}\,|\, y_j\, z_k\,.$$

The proof is straightforward verification of the Poisson brackets.

Using $\mathbf{B}$ and $\mathbf{C}$ (3.9) we determine matrix $\mathbf{A} = \mathrm{diag}(a_1, a_2, a_3)$ and its adjoint $\mathbf{A}^\vee$

$$\mathbf{A} = \frac{1}{2}\mathrm{tr}\,(\mathbf{B}) \,-\mathbf{B} \qquad \text{and} \qquad \mathbf{A}^\vee = \frac{1}{4}(\mathbf{C}^2 - \mathbf{B}^2)\,. \tag{3.11}$$

In these notations let us consider motion of a particle on the surface of the unit sphere

$$S^2 = \{x \in \mathbb{R}^3,\ |x| = 1\} \tag{3.12}$$

under an influence of the fourth order potential. The Hamilton function is equal to

$$\widetilde{H}_1(x,J)=\frac{1}{4}\langle J,J\,\rangle-\langle x,\mathbf{B}x\rangle\Big(\gamma+\delta\langle x,\mathbf{B}x\rangle\Big)+\delta\langle x,\mathbf{A}^\vee x\rangle\,, \tag{3.13}$$

where γ,δ are arbitrary parameters. If $\delta=0$ the function $\widetilde{H}_1(x,J)$ (3.13) is Hamiltonian for the Neumann system. Recall that we can put $|x|=1$ without loss of generality using canonical transformations $x\to a^{-1}x$.

Proposition 4 *Let α^2 and β are the values of the Casimir elements $\mathcal{A}$ and $\mathcal{B}$ (1.4) on $e(3)$ algebra (1.2). If*

$$\gamma=\beta-\alpha^2\mathrm{tr}\,(\mathbf{A})\,,\qquad \delta=\alpha^2 \tag{3.14}$$

the Poisson map $\widetilde{f}_g$ (3.10) identifies the Hamilton function $\widetilde{H}_1$ (3.13) on T^*S^2 with the following Hamiltonian on the algebra $e(3)$ (1.2)

$$H_1=\widetilde{f}_g\Big(\widetilde{H}_1\Big)=H_L-\frac{1}{2}\beta\,\mathrm{tr}\,\mathbf{A}+\alpha^2\,\mathrm{tr}\,\mathbf{A}^\vee\,.$$

This Hamiltonian coincides with the Lyapunov integral H_L (1.7) up to the last two terms depending on the Casimir elements.

The proof is straightforward.

As a result we arrive at the following conclusion: the Steklov-Lyapunov system is equivalent to the potential motion of a point $x\in\mathbb{R}^3$ constrained to the sphere, which belongs to the Stäckel family of integrable systems [19].

For the uniform Stäckel systems we know the Lax matrices, the classical r-matrices, the bi-Hamiltonian description, the Bäcklund transformations, the separated variables, the theta-function solutions and many other facts. The Proposition 4 allows to transfer all these know results concerning to the Stäckel systems onto the Steklov-Lyapunov system directly .

4 The rational Lax matrices

It is known [20], that the Hamiltonian $\widetilde{H}_1(x,J)$ (3.13) is separable in elliptic coordinates $\widetilde{u}_{1,2}$ on the unit sphere S^2 (3.12), which are roots of the standard form

$$e(\lambda)=\frac{(\lambda-\widetilde{u}_1)(\lambda-\widetilde{u}_2)}{\det(\lambda-\mathbf{A})}=\langle x,(\lambda-\mathbf{A})^{-1}x\rangle\,. \tag{4.15}$$

According to [10, 16, 17] the generic 2×2 Lax matrices for the uniform Stäckel system are constructed using Hamiltonian H_1 and the generating function $e(\lambda)$ of the separated variables only

$$\mathscr{L}_r(\lambda)=\begin{pmatrix}-\dfrac{1}{2}\,e_t(\lambda) & e(\lambda)\\[2ex] -\dfrac{1}{2}\,e_{tt}(\lambda)+w(\lambda)\,e(\lambda) & \dfrac{1}{2}\,e_t(\lambda)\end{pmatrix},\qquad \mathscr{A}_r(\lambda)=\begin{pmatrix}0&1\\[2ex] w(\lambda)&0\end{pmatrix}. \tag{4.16}$$

Here $e_t = \{H_1, e\}$ and function $w(\lambda)$ is given by $w(\lambda) = \Big[\phi(\lambda)e(\lambda)^{-1}\Big]_{MN}$, where $\phi(\lambda)$ is a parametric function and $[\xi]_{MN}$ is the linear combinations of the following Laurent projections [16, 17]

$$[\xi]_N = \left[\sum_{k=-\infty}^{+\infty} z_k\lambda^k\right]_N \equiv \sum_{k=-M}^{N} \xi_k\lambda^k\,. \tag{4.17}$$

In our case N=2, $M = 0$ and $\phi(\lambda) = -\delta\lambda - \gamma$. It's easy to prove that the required Lax matrices (4.16) are given by

$$\mathscr{L}_r(\lambda) = \sum_{i=1}^{n=3} \begin{pmatrix} -\dfrac{1}{2}\dfrac{x_i\pi_i}{\lambda - a_i} & \dfrac{x_i^2}{\lambda - a_i} \\ -\dfrac{1}{4}\dfrac{\pi_i^2}{\lambda - a_i} & \dfrac{1}{2}\dfrac{x_i\pi_i}{\lambda - a_i} \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ \delta\lambda + \delta\langle x, \mathbf{B}x\rangle + \gamma & 0 \end{pmatrix}, \tag{4.18}$$

where x_i, π_i are canonical coordinates (3.5) and

$$\mathscr{A}_r(\lambda) = \begin{pmatrix} 0 & 1 \\ -\big(\lambda - \langle x, \mathbf{A}x\rangle\big)\big(\delta\lambda + \delta\langle x, \mathbf{B}x\rangle + \gamma\big) - \delta\big(\langle x, \mathbf{A}x\rangle^2 - \langle x, \mathbf{A}^2x\rangle\big) & 0 \end{pmatrix}.$$

The Poisson bracket relations between the entries of the matrix $\mathscr{L}_r(\lambda)$ are closed into the standard linear r-matrix algebra. The corresponding r-matrix is rational dynamical matrix [16, 17]. Here we present only one relation

$$\left\{-\frac{1}{2}\,e_t(\lambda), e(\mu)\right\}_1' = \frac{e(\lambda) - e(\mu)}{\lambda - \mu}\,, \tag{4.19}$$

which allows us to introduce canonical momenta

$$\widetilde{v}_k = -\,\frac{1}{2}\,e_t(\lambda)\Big|_{\lambda=\widetilde{u}_k} = -\,\frac{1}{2}\sum_{i=1}^{n=3}\frac{x_i\pi_i}{\lambda - a_i}\Big|_{\lambda=\widetilde{u}_k}\,,$$

such that $\{\widetilde{v}_k, \widetilde{u}_j\} = \delta_{kj}$.

Inserting $\lambda = \widetilde{u}_k$ into the determinant of the Lax matrix $\mathscr{L}_r(\lambda)$ (4.18)

$$\det \mathscr{L}_r(\lambda) = \left(-\frac{1}{4}e_t^2 + \frac{1}{2}e\,e_{tt} - w\,e^2\right) = \frac{\widetilde{P}_3(\lambda)}{\det(\lambda - \mathbf{A})}$$

one gets two equations

$$-\frac{1}{4}e_t^2(\widetilde{u}_k) = \frac{\widetilde{P}_3(\widetilde{u}_k)}{\det(\widetilde{u}_k - \mathbf{A})}, \qquad k = 1, 2,$$

where $\widetilde{P}_3$ is generating function of integrals of motion

$$\widetilde{P}_3(\lambda) = \delta\lambda^3 + \gamma\lambda^2 + \lambda\widetilde{H}_1 + \widetilde{H}_2. \tag{4.20}$$

In Hamiltonian variables $\widetilde{v}_k$, $\widetilde{u}_k$ these equations are the separated equations

$$-\widetilde{v}_k^2 = \frac{\widetilde{P}_3(\widetilde{u}_k)}{\det(\widetilde{u}_k - \mathbf{A})}\,, \qquad k = 1, 2, \tag{4.21}$$

whereas in Lagrangian variables $\dot{\widetilde{u}}_k$, $\widetilde{u}_k$ they are equations

$$-\frac{1}{4}\left(\frac{(-1)^{k+1}(\widetilde{u}_1-\widetilde{u}_2)\,\dot{\widetilde{u}}_k}{\det(\widetilde{u}_k-\mathbf{A})}\right)^2=\frac{\widetilde{P}_3(\widetilde{u}_k)}{\det(\widetilde{u}_k-\mathbf{A})}\,,\qquad k=1,2,$$

which may be integrated using Abel-Jacobi inversion theorem.

The second integral of motion $\widetilde{H}_2$ in (4.20) is equal to

$$\widetilde{H}_2(x,J)=-\frac{1}{4}\langle J,\mathbf{A}J\rangle+\langle x,\mathbf{A}^\vee x\rangle\Big(\gamma+\delta\langle x,\mathbf{B}x\rangle\Big)\,. \tag{4.22}$$

As functions on elliptic coordinates $\widetilde{u}_{1,2}$ and momenta $\widetilde{v}_{1,2}$ integrals of motion $\widetilde{H}_{1,2}$ (4.20) on T^*S^2 coincide with the Stäckel integrals (2.11).

So, if conditions (3.14) hold then the Poisson map $\widetilde{f}_g$ (3.10) identifies the second integral of motion $\widetilde{H}_2$ (4.22) on T^*S^2 with the following integral on the algebra $e(3)$ (1.2)

$$H_2(p,M)=\widetilde{f}_g\Big(\widetilde{H}_2\Big)=-H_S+\frac{1}{2}\beta\,\mathrm{tr}\,\mathbf{A}^\vee-\alpha^2\,\det\,\mathbf{A}\,.$$

This is the Steklov integral H_S (1.6) up to the constant terms depending on the Casimir elements.

The same Poisson map $\widetilde{f}_g$ (3.10) identifies standard elliptic coordinates $\widetilde{u}_{1,2}$ (4.15) on the sphere S^2 and the corresponding separated equations (4.21) with the separated variables $u_{1,2}$ (2.7) and the separated equations (2.10) for the Steklov-Lyapunov system proposed by Kötter [7]. It allows us to use the standard finite-band integration technique for the potential motion on the sphere [10] in order to verify the Kötter solution of the Steklov-Lyapunov system in terms of the theta-functions.

Applying the Poisson map $\widetilde{f}_g$ (3.10) to the Lax matrix $\mathscr{L}_r(\lambda)$ (4.18) one gets the rational Lax matrix for the Steklov-Lyapunov system. In [3] the similar Lax matrix was constructed by using the triplet of the vectors x,y,v defined by

$$(M-\mathbf{B}\,p)=x\times y,\qquad p=x\times v\,.$$

5 The Steklov system on $so(4)$ and twisted Poisson map

The bi-Hamiltonian manifold is a smooth manifold $\mathcal{M}$ endowed with a pair of compatible Poisson brackets $\{.\,,.\}_\mathcal{M}$ and $\{.\,,.\}'_\mathcal{M}$. In contrast with the Poisson manifolds we have two opportunities for the action of the mapping $f:\mathcal{M}\to\mathcal{N}$ preserving both Poisson brackets

$$1.\quad f:\ \begin{array}{ccc}\{.\,,.\}_\mathcal{M} & \longleftrightarrow & \{.\,,.\}_\mathcal{N}\\ \\ \{.\,,.\}'_\mathcal{M} & \longleftrightarrow & \{.\,,.\}'_\mathcal{N}\end{array}\,,\qquad 2.\quad f:\ \begin{array}{ccc}\{.\,,.\}_\mathcal{M} & & \{.\,,.\}_\mathcal{N}\\ & \times & \\ \{.\,,.\}'_\mathcal{M} & & \{.\,,.\}'_\mathcal{N}\end{array}\,.$$

In order to distinguish these cases we will say about the Poisson map and the twisted Poisson map at the first and the second cases respectively.

Let us consider the Euler equations on the Lie algebra $so(4)$

$$\dot{s}=s\times\nabla_s H(s,t),\qquad \dot{t}=t\times\nabla_t H(s,t). \tag{5.23}$$

Here vectors s and t are coordinates on $so(4) = so(3) \oplus so(3)$ with the standard Lie-Poisson brackets

$$\bigl\{s_i\,, s_j\,\bigr\}_1^* = \varepsilon_{ijk}\, s_k\,, \qquad \bigl\{s_i\,, t_j\,\bigr\}_1^* = 0\,, \qquad \bigl\{t_i\,, t_j\,\bigr\}_1^* = \varepsilon_{ijk} t_k\,, \tag{5.24}$$

The brackets (5.24) respect two Casimir functions

$$\mathcal{A}^* = |s|^2, \qquad \mathcal{B}^* = |t|^2\,, \tag{5.25}$$

which are therefore integrals of motion for (5.23) in involution with any function on the phase space.

Equations (5.23) describe the motion of a rigid body with elliptic cavities filled with ideal fluid if the Hamilton function is quadratic form

$$H(s,t) = \langle s, \mathbf{A}_1 s\rangle + \langle s, \mathbf{A}_2 t\rangle + \langle t, \mathbf{A}_3 t\rangle.$$

with the special numerical matrices $\mathbf{A}_k$ [12].

In [15] Steklov found the Hamilton function for which equations (5.23) possess a fourth additional integral

$$H = c_1 \widehat{H}_1 + c_2 \widehat{H}_2, \qquad \Bigl\{\widehat{H}_1, \widehat{H}_2\Bigr\}_1^* = 0\,,$$

where $c_{1,2}$ are numerical parameters and

$$\widehat{H}_1 = 2\langle\sqrt{\mathbf{A}^\vee}\, s, t\rangle - \langle s, \mathbf{A}s\rangle\,, \qquad \widehat{H}_2 = \langle t, \mathbf{A}^\vee t\rangle - 2\langle\sqrt{\mathbf{A}^\vee}\, s, \mathbf{A}t\rangle\,. \tag{5.26}$$

In [1] Bobenko established isomorphism between the Steklov system on $so(4)$ and the Steklov-Lyapunov system on $e(3)$.

Proposition 5 *[1] If the phase space $\widehat{M} \simeq so(4)$ is identified with the space $M \simeq e(3)$ by the following linear map*

$$\widehat{f}: \qquad (p, M) \to \left(s = 2p, \quad t = \frac{1}{\sqrt{\mathbf{A}^\vee}}\bigl(M - \mathbf{B}p\bigr)\right)\,, \tag{5.27}$$

where $\mathbf{A}, \mathbf{B}$ are given by (1.5,1.9), then the equations of motion (5.23) for the Steklov system on $so(4)$ coincide with the Kirchhoff equations (1.3) for the Steklov-Lyapunov system on $e(3)$, that is,

$$\bigl\{H_1, .\bigr\}_1 = \bigl\{\widehat{H}_1, .\bigr\}_1^*.$$

Here we changed $p \to 2p$ and $M \to 2M$ in comparison with original map [1] to make formulas for the Poisson pencil slightly more symmetric.

We have to underline that

$$\widehat{H}_1(s,t) \neq \widehat{f}\Bigl(\,H_1(p, M)\Bigr)\,,$$

as for the usual canonical or Poisson transformations. In fact the map $\widehat{f}$ (5.27) gives rise the second Poisson brackets on $so(4)$

$$\bigl\{s_i\,, s_j\,\bigr\}_2^* = 0\,, \qquad \bigl\{s_i\,, t_j\,\bigr\}_2^* = \varepsilon_{ijk}\,\frac{s_k}{\sqrt{\mathbf{A}_{kk}^\vee}}\,, \qquad \bigl\{t_i\,, t_j\,\bigr\}_2^* = \varepsilon_{ijk}\left(\frac{t_k}{\mathbf{A}_{kk}} - \frac{s_k}{\sqrt{\mathbf{A}_{kk}^\vee}}\right)\,, \tag{5.28}$$

whereas inverse map $\widehat{f}^{-1}$ generates the second Poisson brackets on $e(3)$

$$\{M_i,M_j\}_2=\varepsilon_{ijk}\left(\mathbf{A}_{kk}M_k-\frac{\mathbf{C}^\vee_{kk}}{2}p_k\right),\quad \{M_i,p_j\}_2=\varepsilon_{ijk}\frac{\mathbf{B}_{kk}}{2}p_k,\quad \{p_i,p_j\}_2=\varepsilon_{ijk}\frac{p_k}{2}\,. \tag{5.29}$$

Here $\mathbf{C}^\vee$ is the cofactor matrix to matrix $\mathbf{C}$ (1.5).

Proposition 6 *The brackets $\{.,.\}_1$ (1.2) and $\{.,.\}_2$ (5.29) on the manifold $e(3)$ and brackets $\{.,.\}^*_1$ (5.24) and $\{.,.\}^*_2$ (5.28) on the manifold $so(4)$ are compatible.*

The proof consists of the verification that every linear combination of these brackets is still a Poisson bracket on $e(3)$ and $so(4)$ respectively.

This Proposition allows us to say that the mapping $\widehat{f}$ (5.27) is a twisted Poisson map, which identifies two bi-Hamiltonian manifolds $e(3)$ and $so(4)$ such that

$$\begin{tikzcd} \mathscr{P}_1 \arrow[dr, <->] & \mathscr{P}_1^* \arrow[dl, <->] \\ \mathscr{P}_2 & \mathscr{P}_2^* \end{tikzcd} \qquad \text{instead of} \qquad \begin{tikzcd} \mathscr{P}_1 \arrow[r, <->] & \mathscr{P}_1^* \\ \mathscr{P}_2 \arrow[r, <->] & \mathscr{P}_2^* \end{tikzcd} \qquad \text{as for the usual Poisson map.}$$

The bi-Hamiltonian systems are defined on the bi-Hamiltonian manifolds by the coefficients of the Casimir functions of the Poisson pencil $\{.\,,.\}_\lambda=\{.\,,.\}_1-\lambda\{.\,,.\}_2$. In our case on the bi-Hamiltonian manifold $e(3)$ cubic polynomial $P_3(\lambda)=|\ell|^2$ (2.2)

$$P_3(\lambda)=\sum_{k=0}^{3}\mathcal{H}_k\lambda^k=\mathcal{A}\lambda^3+(\mathcal{B}-\mathcal{A}\mathrm{tr}\,\mathbf{A})\lambda^2+\frac{H_1}{4}\,\lambda+\frac{H_2}{4}\,. \tag{5.30}$$

is a Casimir of the Poisson pencil

$$\mathscr{P}_\lambda=\mathscr{P}_2-\lambda\mathscr{P}_1,\qquad \mathscr{P}_\lambda\,dP_3(\lambda,p,M)=0\,. \tag{5.31}$$

Here $\mathscr{P}_1$ and $\mathscr{P}_2$ are the Poisson tensors associated with the brackets $\{.,.\}_1$ (1.2) and $\{.,.\}_2$ (5.29).

On the bi-Hamiltonian manifold $so(4)$ polynomial

$$\widehat{P}_3(\lambda)=\widehat{f}\Big(P_3(\lambda)\Big)=\sum_{k=0}^{3}\widehat{\mathcal{H}}_k\lambda^k=\mathcal{A}^*\,\lambda^3+\widehat{H}_1\lambda^2+\widehat{H}_2\,\lambda-\det\mathbf{A}\,\mathcal{B}^* \tag{5.32}$$

is a Casimir of the Poisson pencil

$$\mathscr{P}^*_\lambda=\mathscr{P}^*_1-\lambda\mathscr{P}^*_2,\qquad \mathscr{P}^*_\lambda\,d\widehat{P}_3(\lambda,s,t)=0\,. \tag{5.33}$$

Here $\mathscr{P}^*_1$ and $\mathscr{P}^*_2$ are the Poisson tensors associated with the brackets $\{.,.\}^*_1$ (5.24) and $\{.,.\}^*_2$ (5.28).

As usual, coefficients of the polynomial $P_3(\lambda)$ (5.30) form a bi-Hamiltonian hierarchy on $e(3)$ starting from a Casimir $\mathcal{H}_0$ of $\mathscr{P}_2$ and terminating with a Casimir of $\mathscr{P}_1$

$$\mathscr{P}_2d\mathcal{H}_0=0,\qquad \mathscr{P}_2d\mathcal{H}_{i+1}=\mathscr{P}_1d\mathcal{H}_i,\qquad \mathscr{P}_1d\mathcal{H}_3=0\,.$$

Coefficients of the polynomial $\widehat{P}_3(\lambda)$ (5.32) form the similar Lenard chain on $so(4)$. Therefore, the Steklov-Lyapunov system on $e(3)$ and the Steklov system on $so(4)$ are the Gelfand-Zakharevich systems [4].

It is known that the Gelfand-Zakharevich systems admit the Stäckel separation of variables (see [5] and references within). However in our case tensors $\mathscr{P}_{1,2}$ and $\mathscr{P}^*_{1,2}$ are degenerate and, therefore, the corresponding separated variables could be obtained after a suitable reduction of the Poisson structures [5, 9] only.

A natural way to do that is to fix the values of the Casimir functions of $\mathscr{P}_1$ or $\mathscr{P}^*_1$. However, in the our case each Poisson tensor $\mathscr{P}_1$ ($\mathscr{P}^*_1$) or $\mathscr{P}_2$ ($\mathscr{P}^*_2$) can be properly restricted to a corresponding symplectic leaf, but the other tensor does not restrict to same leaf. So, a quite general reduction technique given by the Marsden-Ratiu theorem [5, 9] have to be applied in this case.

In fact we make the necessary reduction when we introduce the chart (x, J) related to (p, M) by the map $\widetilde{f}_g$ (3.10). Namely, composition of the maps $\widehat{f}$ (5.27) and $\widetilde{f}_g$ (3.10) gives rise to the second Poisson brackets on the second sample of $e(3)$ (3.1)

$$\{J_i, J_j\}'_2 = \varepsilon_{ijk} a_k J_k, \qquad \{J_i, x_j\}'_2 = \varepsilon_{ijk} a_j x_k - \mathbf{C}_{ii} \frac{x_1 x_2 x_3}{|x|^2} \frac{x_j}{x_i}, \qquad \{x_i, x_j\}'_2 = 0\,. \tag{5.34}$$

The Casimir of the Poisson pencil $\mathscr{P}'_1 - \lambda \mathscr{P}'_2$ coincides with the cubic polynomial $\widetilde{P}_3(\lambda)$ (4.20)

$$\widetilde{P}_3(\lambda) = \delta\lambda^3 + \gamma\lambda^2 + \widetilde{H}_1\lambda + \widetilde{H}_2\,, \qquad \mathscr{P}'_\lambda\, d\widetilde{P}_3(\lambda, x, J) = 0\,.$$

This polynomial Casimir includes two numerical constants γ and δ instead of the Casimir functions and, therefore, the corresponding symplectic foliation is trivial. According to [18], in this case the separated variables (4.15) are the solutions of the following system of algebraic equations

$$\frac{\partial}{\partial\gamma}\, \widetilde{P}_3(\lambda, x, J) = 0\,, \qquad \frac{\partial}{\partial\delta}\, \widetilde{P}_3(\lambda, x, J) = 0\,.$$

These separated variables have to coincide with the eigenvalues of the reduced Nijhenuis tensors $\mathscr{N} = \mathscr{P}_1\mathscr{P}_2^{-1}$ or $\widehat{\mathscr{N}} = \widehat{\mathscr{P}_1}\widehat{\mathscr{P}_2}^{-1}$. As sequence these variables have to be in the involution with respect to both Poisson brackets similar to the integrals of motion. We can check this property of the Kötter separated variables (2.7) without reduction of the Poisson tensors.

Proposition 7 *The vectors*

$$x = \frac{(M - \mathbf{B}p) \times p}{|(M - \mathbf{B}p) \times p\,|} \qquad \text{and} \qquad \widehat{x} = \frac{\sqrt{\mathbf{A}^\vee} t \times s}{\sqrt{\mathbf{A}^\vee} t \times s},$$

and, therefore, zeroes of the standard equations

$$\big\langle x, (\lambda - \mathbf{A})^{-1} x \,\big\rangle = 0, \qquad \text{and} \qquad \big\langle \widehat{x}, (\lambda - \mathbf{A})^{-1} \widehat{x} \,\big\rangle = 0$$

are in the involution with respect to both Poisson brackets on $e(3)$ and $so(4)$ respectively.

This Proposition has a straightforward consequence. The roots $\widehat{u}_{1,2} = \widehat{f}(u_{1,2})$ of the non-diagonal entry of the corresponding rational Lax matrix (4.16)

$$\widehat{e}(\lambda) = \frac{(\lambda - \widehat{u}_1)(\lambda - \widehat{u}_2)}{\det(\lambda - \mathbf{A})} = \sum_{i=1}^{n=3} \frac{\widehat{x}_i^2}{\lambda - a_i} = 0 \tag{5.35}$$

are in the involution $\{\widehat{u}_1,\widehat{u}_2\}^*_1=\{\widehat{u}_1,\widehat{u}_2\}^*_2=0$ on $so(4)$ and, according to Proposition 5, they satisfy to the following equations

$$\frac{d}{dt}\widehat{u}_1=\frac{2\sqrt{\widehat{P}_3(\widehat{u}_1)\det(\mathbf{A}-\widehat{u}_1)}}{\widehat{u}_1-\widehat{u}_2},\qquad \frac{d}{dt}\widehat{u}_2=-\frac{2\sqrt{\widehat{P}_3(\widehat{u}_2)\det(\mathbf{A}-\widehat{u}_2)}}{\widehat{u}_1-\widehat{u}_2}\,. \tag{5.36}$$

Hence $\widehat{u}_{1,2}$ are the separated variables for the Steklov system on $so(4)$. These variables are poles of the eigenvector $\widehat{\mathbf{\Psi}}(\lambda)$ of the Lax matrix $\widehat{\mathscr{L}_e}(\lambda)$

$$\widehat{\mathscr{L}_e}(\lambda)_{ij}=\varepsilon_{ijk}\sqrt{\lambda-a_k}\left(s_k+\lambda^{-1}\sqrt{\mathbf{A}^\vee_k}t_k\right)$$

with the following dynamical normalization

$$\widehat{\boldsymbol{\alpha}}=|\sqrt{\mathbf{A}^\vee}\,t\times s\,|^{-1}\,\mathbf{W}\,s.$$

The conjugated momenta $\widehat{v}_{1,2}$ are easy defined by the diagonal entry of the corresponding rational Lax matrix (4.16)

$$\widehat{v}_j=-\frac{1}{2\widehat{u}_j}\,\{\widehat{H}_1,\widehat{e}(\lambda)\}\Big|_{\lambda=\widehat{u}_j}\,,\qquad \{\widehat{v}_j,\widehat{u}_j\}^*_1=\delta_{jk}$$

after calculation of the relation (4.19) with respect to the second Poisson brackets (5.34). The corresponding separated equations are equal to

$$\widehat{v}_j^2+\frac{\widehat{P}_3(\widehat{u}_j)}{\widehat{u}_j^2\,\det(\widehat{u}_j-\mathbf{A})}=0\,,\qquad j=1,2,$$

such that the Stäckel matrix

$$\widehat{\mathcal{S}}=\left(\begin{array}{cc}u_1^2 & u_2^2\\ u_1 & u_2\end{array}\right) \tag{5.37}$$

is another block of the same Brill-Noether matrix $\mathcal{U}_\mathcal{C}$ (2.13). According to [17] the Stäckel systems associated with the different blocks of a common Brill-Noether matrix are related by canonical transformation of the time.

In contrast with the Steklov-Lyapunov system for the Stäckel matrix (5.37) associated with the Steklov system we do not know how to introduce coordinates in which equations of motion are the Newton equations.

6 Concluding remarks

The first result in this paper is that, starting with the Kötter separated variables we construct the Poisson map which transforms the Steklov-Lyapunov case of the Kirchhoff equations to the Newton equations on the sphere. A natural way to construct such maps is to identify the separated variables and the corresponding separated equations.

The separation of variables for the Steklov-Lyapunov system on $e(3)$ and for the Steklov system on $so(4)$ is discussed in framework of the Sklyanin method and in the bi-Hamiltonian approach. The main unsolved questions are how to construct a suitable normalization of

the Baker-Akhiezer function and a suitable reduction of the degenerate Poisson tensors. To solve these questions on this example we could to construct unknown separated variables for the Clebsch system on $e(3)$ and the Shottky-Manakov system on $so(4)$. As above, the bi-Hamiltonian structure of these manifolds are defined by the similar linear change of variables proposed in [1].

As a last remark we recall that in [3] there are separated variables, the elliptic and rational Lax matrices and the compatible Poisson tensors for the multi-dimensional Steklov systems. In our opinion these results deserve further investigation in framework of the separation of variables theory and of the general reduction theory for bi-Hamiltonian manifolds.

The research was partially supported by RFBR grant 02-01-00888.

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